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Teaching Of Calculus For Students' Conceptual Understanding

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Abstract

Mathematicians consider calculus to be the root of higher level mathematics taught at university. Thus a last course in school or a first course in university is usually calculus. Students' conceptual understanding of calculus, however, remains an issue. Drawing on experiences in teaching for a conceptual understanding of calculus, I argue that students can develop their conceptual understanding of calculus by exploring graphs and functions and by developing intuitive understandings of limit and continuity.

Introduction

Two kinds of calculus courses are discussed in this paper. An informal course in calculus is one which emphasizes an intuitive understanding of different topics such as limits, continuity, rate of change and area under the curve, but does not make tight definitions or give formal proofs. Formal calculus, on the other hand, starts with definitions, rules and proofs and then allows for lots of problems. If, for example, $\lim_{x \to 0} \frac{\sin x}{x} = 1$ is stated and proved towards the beginning of a course I would call it part of formal calculus. On the other hand, if the value of $\frac{\sin x}{x}$ as $x \to 0$ is explored with the help of graphs I would call it informal.

The idea of formal and informal calculus is closely connected with the notion of "concept definition", "formal concept definition", and "concept images" developed by Tall and Vinner (1981). They distinguish among these terms as follows: A concept definition may be constructed by a student in the form of words to understand a mathematical concept which may or may not be accepted by the mathematical community as a part of formal theory. For example, if a student understands limit concept such as $S_n \to s$ as S_n can be made as close to s as we please by taking sufficiently large n, but S_n cannot be equal to s, it can be considered as a concept definition. When the ε, δ symbols are introduced to define limits (i.e. epsilon-delta definition of limit), the definition, then, is a formal concept definition of limit. Concept image, on the other hand, is the visualization of a limit concept based on some particular examples such as the limit of the sequence .9, .99, .999, ... is 1. Usually, students at this stage have only an intuitive meaning of limit. Tall and Vinner (1981) define concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Since all these cognitive structures are not necessarily coherent with each other, concept images can be quite different from the formal concept definition.

Informal calculus when associated with the theory of Tall and Vinner consists of concept images and concept definitions. Formal calculus, on the other hand, emphasizes on formal concept definition. An informal course in calculus emphasizes the students' intuitive and conceptual understanding of different calculus topics. It is important to stress that the beginners in calculus should not be pushed towards a formal and rigorous approach without having such experiences. The history of mathematics shows us that rigor and proof usually followed observation, analogy, induction and intuition (Kleiner, 1988). The history of calculus reminds us that rigorous proofs in analysis were made only by nineteenth century mathematicians like Cauchy and Bolzano even though eighteenth century mathematicians like Leibniz and Newton had a deep understanding of the properties of basic concepts of analysis (Grabiner, 1974).

How is Calculus Taught?

Broadly speaking, there are two ways of teaching calculus. First, there is the traditional way in which teachers give rules and students apply them to solve routine problems without really understanding what they are doing. If they are asked why they do so, the immediate answers would be "this is how we are taught" or "this is how you can get the right answer". With this approach students rarely question why the rule works and where the rule comes from. Skemp (1976) has said this approach leads to instrumental understanding, that is to say the ability to apply rules without reasons. An instrumental way of teaching does not produce creative thinkers. Unfortunately, students' understanding of calculus all over the world is mostly instrumental (Ferrini-Mundy & Gaudard, 1992; Gooya, 1988; Orton 1983a, b; Tall & Vinner, 1981). It is interesting that when students in Orton's (1983a) study were asked to give the reasons for following a particular rule to solve a problem some of them answered, "Well, that's the way you are taught to do it; the other way is what you get onto later." (p. 9). Presumably, this is not the type of responses we want from calculus students.

Skemp (1976) argues that teaching should emphasize developing relational understanding. Students with a relational understanding not only know what to do and how to do it, but also can explain what they are doing. I agree with Skemp that calculus teaching should focus on developing relational understanding. However, relational understanding of calculus does not automatically imply that students should understand the intuitive meaning of the terms such as limit, derivative and integration before going to solve the problems related to these topics. Relational understanding can also be developed in the domain of formal calculus without ever going to informal calculus. I prefer to use the term conceptual understanding instead of relational understanding to include the informal and intuitive aspects of calculus. Students can demonstrate conceptual understanding of calculus by explaining the meaning of concepts with the help of graphs, diagrams, or examples from everyday situations.

The second method of teaching calculus is based on exploratory, intuitive approaches to developing the meanings of different concepts in calculus such as limit, continuity, differentiation, and integration. Implicit in this method is the assumption that students can have an intuitive understanding of concepts and rules before going to formal definitions. At this stage students can explain concepts and rules on the basis of some particular examples but cannot give formal definitions. Students prove these rules mathematically only at a later stage. The teacher in the classroom serves as a facilitator rather than a teller. Students create and recreate their knowledge working individually or in groups.

Many authors argue that the second method of teaching is needed if students are to gain a conceptual understanding of calculus (Ferrini-Mundy & Gaudard, 1992; Gooya, 1988; Koirala & Koe, 1992; National Council of Teachers of Mathematics, 1989; Orton 1983a, 1983b; Tall & Vinner, 1981). Yet the teaching of calculus has not moved along this line very far.

How should Calculus be Taught?

Mainly based on my experience of teaching calculus and an inquiry of my own teaching for several years I will argue that students' conceptual understanding of calculus can be enhanced by emphasizing graphs and functions and intuitive understanding of limits. I also agree with the above-mentioned authors that the traditional method of teaching calculus hinders students to understand calculus conceptually, and so exploratory and intuitive approaches to teaching should be used.

In my own teaching in the past, I emphasized on students' conceptual understanding. Basically, I used three ways to enhance students' conceptual understanding of calculus.

<u>More time for graphs and functions</u>

In the beginning of all my calculus classes, the primary focus was on foundation topics such as graphs, functions, and limits. I provided sufficient time for these topics so that students had a good sense of the meanings of these terms. Many students demonstrated understanding of even a cumbersome theorem such as $\lim_{x \to 0} \frac{\sin x}{x} = 1$ by first interpreting the graph of $\frac{\sin x}{x}$ as $x \to 0$. I found that understanding of these topics was helpful for students to conceptually understand differentiation and integration in later days. On the other hand, many teachers and instructors do not deal or deal very little with graphs and functions in their formal calculus course. Data from the Second International Mathematics Study (SIMS) indicate that in countries like the United States and Canada, a large number of teachers did not employ the use of graphs to develop limits and continuity (Dirks, Robitaille, & Leduc, 1989). If calculus teachers rush to evaluate limit problems such as $\lim_{x \to \infty} \frac{x+3}{x^2+1}$, $\lim_{x \to 0} \frac{1-\cos x}{x}$ without discussing $\lim_{x \to \infty} \frac{1}{x} = 0$ through the graph of $\frac{1}{x}$, how can we ascertain that students can understand infinite limits even if they evaluate the limits in a mechanical way?

Teachers and instructors find that calculus courses in most schools and colleges are designed to cover many topics in a limited time. Since all these topics are to be finished for the examination, a significant amount of time cannot be given for graph work. Although, more calculus topics may be included in the syllabus if graphs and functions are not emphasized, a good conceptual understanding of fewer topics may be better than a superficial understanding of many topics. Also, the inclusion of more topics and problems in a calculus course does not necessarily imply that students will understand. This does not mean that many topics from the present calculus course should be omitted. If possible, more time should be added for the teaching of calculus, but this is unlikely both in schools and colleges or universities because there are other equally important topics to be taught. Thus the best way of overcoming this problem is to omit some of the existing drill-work so that the place can be made for graphs and functions.

Fortunately, the advent of graphic calculators and computers has made it easier for many students to draw and interpret different types of graphs. Students who use calculators and graphic software packages can spend more time in interpreting graphs rather than constructing them. This shift from constructing graphs to interpreting graphs provides more opportunities for students to develop conceptual understanding of functions, limits, continuity, differentiation, and integration. Orton, for his problem about the tangent line, recommended the use of electronic calculators to study the slope of a tangent line at a point P through the slopes of secant PQ where Q takes on different positions on the curve. The point Orton was making was the fact that the calculation could be done more quickly and accurately. But the technology today has developed even further, bringing not only the possibility of the quick and precise calculation of ratios but also the visualization of the secant line approaching towards a tangent line through the use of graphic software package in a computer.

Intuitive understanding of limits

Many students have difficulty understanding calculus because they have problems understanding the concept of limits. Since differentiation and intĕgration are both based on limits, it is important that students comprehend the meaning of limit before they go on to formal treatments in differentiation and integration. Perhaps the first stage in doing so is to explore the students' intuitive understanding of infinity. Many students may have misconceptions about infinity. These should not be taken lightly because we know from the history of mathematics that even the greatest mathematicians had problems understanding infinite processes. One of the main reasons why calculus was invented only in the eighteenth century was the lack of understanding of infinite processes before Newton and Leibniz. If the concept of infinity had been well understood, calculus might have been invented by the ancient Greeks when Archimedes was able to find area under a parabolic segment.

Fischbein, Tirosh, and Melamed (1981) have claimed that many difficulties arising in learning mathematics are the product of poor, incorrect, or inadequate intuitive interpretations. A number of examples which may be appropriate to understand the meanings of infinity for Grade 8 and above are discussed by Fischbein et al. Perhaps all students coming to take a calculus course should have already taken an algebra course dealing with infinite geometric series. However, the calculus teacher should not assume that students know the concept of infinity very well in advance. I usually opened my calculus course with a discussion of infinity. I found that the discussion was useful and well linked to the concept of limit. Some of the examples which my students and I discussed in the classrooms were the following.

1. Is it possible to measure the growth of a plant in the interval of 1 second?
2. If a man drinks half a cup of tea at first time and continues to drink half of the remainder will he be able to finish all the tea in the cup?
3. A three sided regular polygon is inscribed in a circle. Can the area of the inscribed polygon be made equal to the area of the circle by increasing the number of sides of the polygon?
4. A square is drawn by joining the midpoints of the consecutive sides of a square. If we continue this process again and again, is it possible to get a square whose area is zero?

These examples are very useful to develop students' intuitive understanding of limit. The notion of an inscribed polygon and circle provides students with the opportunity of comparing the area of the inscribed polygon and the area of the circle as the number of sides (n) of the polygon increases from 3 to 4, 5, 6, ... For a few steps, students will be interested in sketching the figure, but very soon they will realize that the area of the inscribed polygon approaches the area of the circle as $n \to \infty$. In other words, students understand that a circle is the limiting figure of a regular polygon as the number of sides of the polygon increases infinitely.

The intuitive idea of limits can be developed not only through the examples as above but even earlier by paper folding and curve stitching. A series like $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ can be obtained very easily by paper folding. (In fact this series is also produced in example 2 above). Other examples which we discussed and led to calculus were steepness of a road (up and down), rate of change of profit as a function of cost, and approximation of area by counting small squares. These examples looked quite simple but were very powerful in developing students' conception of limit.

Moving slowing towards a formal approach

Although I have argued above for more time for graphs and functions and an intuitive understanding of limit, it is nevertheless essential that a calculus course in college or university should provide formal treatment of the topics limit, differentiation, and integration. This content is not easy and can be even more complicated when the students have already taken an informal approach to calculus by graphs and intuition (Tall & Vinner, 1981; Williams, 1991).

Tall and Vinner (1981) observed that the students' concept images of limit and continuity conflicted with the formal concept definition. When 70 students were asked to explain the meaning of $\lim_{x \to 1} \left(\frac{x^3 - 1}{x - 1} \right) = 3$ and to write the formal definition of $\lim_{x \to a} f(x) = 1$, only four students could give the correct formal definition of limit. However, the majority of the students including those who could not write the formal definition explained the meaning of $\lim_{x \to 1} \left(\frac{x^3 - 1}{x - 1} \right) = 3$. It shows that once the students develop a concept image they heavily rely on it even after they are exposed to formal definition.

Similarly, out of 41 students arriving at a university in England 35 claimed that $f(x) = \frac{1}{x} (x \neq 0)$ is discontinuous because the graph of this function has a gap and the function is not defined at the origin, even though these students had already met such a formal definition of continuity as f is said to be continuous at a if $\lim_{x \to a} f(x) = f(a)$. Tall and Vinner argue that if students had used the formal definition they would definitely have come to the conclusion that $f(x) = \frac{1}{x} (x \neq 0)$ is a continuous function.

Similarly, Williams (1991) was not able to bring a "cognitive change" in college students who were accustomed to an informal approach to calculus, even after presenting the students with an alternative formal model of limit in five sessions over a period of seven weeks. Williams believes that with careful and explicit instructions and with a rich variety of limit models, over a longer period of time, students can be helped in attaining formal approach to limits. He also believes that students' "prior knowledge of graphs and functions must be deconstructed, to expose the underlying assumptions that formal definitions attempt to address" (p. 235). Although, Williams did not explicitly clarify what he meant by "deconstruction of graphs and functions" he seemed to be convinced that students' over-faith in graphing as a means of understanding and justifying the limit concept and doing limit problems made it difficult for them to appreciate the need of a more formal approach to limit. Nevertheless, he was able to create some "cognitive conflict" in students' limit concepts, which was the required step to bring cognitive change. I have to admit that cognitive change was difficult even in my own teaching because students expected that everything in calculus could be intuitively understood by drawing and interpreting graphs and functions. However, by providing carefully selected examples, students realized that not everything in calculus was possible only by intuitive understanding. The most encouraging point for me was that cognitive change was possible without destroying students' existing limit concepts. Hence, I would prefer to use the phrase "cognitive extension" rather than cognitive change.

My main strategy for cognitive extension was to introduce conflict and controversy deliberately into the classroom. In a way my strategy for cognitive extension was similar to what Hewson et al. (1992) suggest in their paper. According to them, learning can take place easily if the learner finds the new knowledge to be intelligible (knows what it means), plausible (believes to be true), and fruitful (finds it to be useful). If this can be done in calculus class students' existing concepts of graphs, functions and limits can be reconstructed. For example, if we ask our students to find the sum of series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$, it is less likely that they can find this sum by intuition. Some students may try to find the sum by adding two consecutive numbers of the series, for instance, $\frac{1}{3}+\frac{1}{9}=\frac{4}{9}$, $\frac{1}{27}+\frac{1}{81}=\frac{4}{81}$, ..., and repeating the process for several times. However, this method is cumbersome. In such a situation, they will appreciate the need of the construction of a formula for infinite geometric series

$1+r+r^2+\ldots=\frac{a}{1-r}(r<1)$, but it is not necessary for their existing understanding of limit concept to be destroyed. Similarly, if we turn to the example provided by Tall and Vinner about the continuity of the function $f(x)=\frac{1}{x}(x\neq 0)$, it can be easily shown by its graph that it is a continuous function. At this point students should be encouraged to evaluate the continuity of the function both by graphical approach and formal approach. Students should discuss in pairs and groups about the effectiveness of these approaches. What we really need to emphasize is the link between informal and formal approaches so that the students can appreciate formal approaches to limit, continuity, derivative, or integral not necessarily destroying their informal intuitive understanding.

Conclusion

An introductory calculus course should be informal, intuitive, and conceptually based mainly on graphs and functions. Examples and activities in a calculus book should be selected from the real world as far as possible so that students can appreciate that calculus has something to do with their daily life. Formulas and rules should not be given as granted but they should be carefully developed intuitively on the basis of students' previous work in mathematics and science. Some simple application problems with rates of change, maxima and minima, and area under curves should be included in the textbook. The major focus of this course for students in schools should not be to obtain the college credits but to be able to appreciate the importance of calculus both in their daily life and in the study of higher mathematics.

Students should be given the opportunity to benefit from the use of advanced calculators and computer graphic packages while doing graph work. A good exposure in graph work, mainly interpretation, is a necessary condition for the students in calculus, but this is not sufficient in itself. Links between graphs and their functions and the rate of change, as well as area under graphs should be established through a lot of experiences. Discussions in small groups as well as in the whole group can help in eradicating students' misunderstandings. Because these processes take a long time, students and teachers should not rush towards formal calculus until the students understand these concepts.

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